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Book Reviews

BOOK REVIEW EDITOR: WALTER VAN ASSCHE

Books

M. ATTEIA, *Hilbertian Kernels and Spline Functions*, Studies in Computational Mathematics, Vol. **4**, North-Holland, 1992, xii + 386 pp.

There are many different aspects to the study of spline functions. Splines enjoy a prominent place in classical approximation theory, they are remarkably useful in practical numerical analysis, and statisticians have found them useful. Atteia's book considers spline functions from a more abstract and unifying point of view as solutions of certain variational problems. This approach is typified by a theorem in Chapter 3 which tells us that if $u: X \to Y$ and $v: X \to Z$ are continuous linear functions between Hilbert spaces for which $u(\ker v)$ is closed in Y and $\ker u \cap \ker v = \{0\}$, then for each $z \in Z$ there is a unique $s \in X$ such that

$$\|us\|_Y = \min\{\|ux\|_Y : x \in X, vx = z\}.$$

This s is an interpolating spline function which generalizes the familiar classical versions.

The abstract approach can be traced back at least as far as the paper by Golomb and Weinberger [1]. It has been popular in the French school with Atteia as a prominent contributor. This book draws together many works to demonstrate the importance of Hilbertian kernels (known more commonly as reproducing kernels) and Hilbert spaces for the study of splines.

Hilbertian kernels arise as "typical" representations of certain function spaces. Given a Hilbert space H of (real or complex valued) functions on some domain Ω which are bounded in a certain sense, then for each $t \in \Omega$ there is a function $R(\cdot, t) \in H$ which has the reproducing property

$$f(t) = (f, R(\cdot, t)) \qquad \text{for all} \qquad f \in H.$$

Such a Hilbert space is termed a reproducing kernel Hilbert space, and R is its Hilbertian kernel. It turns out that in many cases the spline function s is obtained from the Hilbertian kernel R associated with X as a linear combination

$$s = \sum \lambda_i R(\cdot, t_i).$$

This is essentially a projection-type principle. The coefficients can be determined from the obvious matrix equation, though, numerically, this may be ill-conditioned. Multivariate splines, an area of much current research, can also be obtained from the same projection principle.

The author has tried to ensure that this work is "as far as possible self-contained" as well as being a reference, and so he provides a comprehensive introduction to Hilbertian kernels and abstract splines which makes up nearly two-thirds of the book. The first four chapters detail Atteia's approach to splines (and interpolation in general) and include many examples to orient the reader already familiar with the classical approach, e.g., Lagrange and spline interpolants. There is also a nice section about interpolating on an infinite set of nodes, as well as a chapter examining the convergence of splines.

0021-9045/94 $6.00

The reader who is not reasonably familiar with functional analysis will need to have copies of standard texts close to hand. Nevertheless, the chapter on Hilbertian kernels is one of the very few monograph sources in any context on this important subject.

The final third of the book is devoted to extending the introductory material. Here the reader is exposed to recent work on spline functions on convex sets in which the basic problem is minimizing $\|ux\|_Y$ over $\{x \in X: vx \in C\}$ where $C \subset Z$ is a closed convex set. Also covered are box-splines, B-splines, and simplicial splines, together with an interpolated chapter on spline manifolds which occur in minimizing quadratic energy functionals found in acoustics, elasticity, and the like.

Grace Wahba [2] opines "*... that the effort to master the basic properties of [reproducing kernel Hilbert spaces]... will be worth the effort.*" Unfortunately the technical deficiencies of the book, having to do with editing and the lack of an index, make the effort harder than is necessary. However, the material Atteia presents is important, well worth knowing, and naturally appealing.

References

1. M. Golomb and H. Weinberger, *Optimal approximation and error bounds*, *in* "On Numerical Approximation" (R. Langer, Ed.), pp. 117–190, Univ. of Wisconsin, Madison, 1958.
2. G. Wahba, "*Spline Models for Variational Data*," SIAM, Philadelphia, 1986.

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D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and Its Applications (East European Series) Vol. **61**, Kluwer Academic, 1993, xvii + 740 pp.

This is a large book about inequalities in analysis. It is 740 pages long, has over 1000 references, and is broken into 30 more-or-less self contained chapters. These chapters have titles like "Bessel's Inequality," "Norm Inequalities," and "Shannon's Inequality." The typical chapter consists of many inequalities, a few proofs, some discussion, some history and references. The intention is that the chapters can be read by themselves and while there is some cross referencing, it is kept to a minimum.

In the introduction the authors ask: "*... why another book on inequalities?*" and offer two answers: "*First, there are many recent refinements of these inequalities and connections between the various ones that have not been noticed before. Secondly, many inequalities are proved over and over, mainly because mathematicians are unaware of the previous history....*" They add: "*It is our purpose to address these issues by attempting a comprehensive study of the publications devoted to inequalities.*"

This is an ambitious book. It should also be a useful book, primarily as a "handbook" of inequalities, a reference book for the working mathematician, engineer, or physicist who actually needs an inequality. While it would be possible to study out of this book, I suspect that this will not be its major use. It is possible that this book will become a standard reference. The only serious competition is "Inequalities" by Hardy, Littlewood, and Pólya, and this book, while it has aged well, is 60 years old. I also suspect that it will take time to measure the real utility of the book. There is a wealth of material, but it takes some practice to find things. A serious shortcoming is the lack of an index. The only index is a names index, there is not even an extended table of contents. A second serious shortcoming is the price. At $285 U.S., it is unlikely to appeal to the casual user.

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